Vectorized Gaussian Input/ Output

David Carcamo

January 2023

1 The Result

The mutual information rate for M input time series and Q output time series takes the form

$$m(I;O) = -\frac{1}{4\pi}\int d\omega \ln\left(\frac{\det \widetilde{Z}(\omega)}{\det \widetilde{A}(\omega)\det \widetilde{B}(\omega)}\right) \tag{1}$$

Where $\widetilde{Z}(\omega)$ is a 2×2 block matrix of the form

$$\widetilde{Z}(\omega) = \begin{bmatrix} C^{\mathbf{I},\mathbf{I}}(\omega) & C^{\mathbf{I},\mathbf{O}}(\omega) \\ C^{\mathbf{O},\mathbf{I}}(\omega) & C^{\mathbf{O},\mathbf{O}}(\omega) \end{bmatrix} \tag{2}$$

The elements of matrix $C^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\omega)$ are $C^{\alpha_j,\beta_k}(\omega)$ which are the correlations between the jth α time series and the kth β time series, where α and β represent either the input or output time series. $\widetilde{A}(\omega)$ is the upper left hand matrix of $\widetilde{Z}(\omega)$ or in terms of the correlation functions the $\widetilde{A}(\omega)_{j,k}$ element is $C^{I_j,I_k}(\omega)$. Similarly, $\widetilde{B}(\omega)$ is the lower right hand matrix of $\widetilde{Z}(\omega)$.

2 One input, two outputs

In the case of one input and two outputs our total spectral density matrix takes the form

$$\begin{bmatrix} C_{II}(\omega) & C_{IO_1}(\omega) & C_{IO_2}(\omega) \\ C_{O_1I}(\omega) & C_{O_1O_1}(\omega) & C_{O_1O_2}(\omega) \\ C_{O_2I}(\omega) & C_{O_2O_1}(\omega) & C_{O_2O_2}(\omega) \end{bmatrix} \tag{3}$$

Using the formula above for the information rate

$$m(I;O) = -\frac{1}{4\pi}\int d\omega \ln\left(1 + \frac{1}{C_{II}(\omega)}\frac{2Re\left[C_{IO_1}(\omega)C_{IO_2}(\omega)^{\dagger}C_{O_1O_2}(\omega)\right] - C_{O_1O_1}(\omega)|C_{IO_2}(\omega)|^2 - C_{O_2O_2}(\omega)|C_{IO_1}(\omega)|^2}{C_{O_1O_1}(\omega)C_{O_2O_2}(\omega) - |C_{O_1O_2}(\omega)|^2}\right) \tag{4}$$

As a sanity check, if we remove all correlations involving the second output with either the input or the first output, the formula reduces back to the one output formula.

$$m(I,O) = -\frac{1}{4\pi}\int d\omega \ln\left(1 - |C_{I,O}(\omega)|^2/C_{I,I}(\omega)C_{O,O}(\omega)\right) \tag{5}$$

3 Standard Case

First lets start with finding the mutual information for one Gaussian distributed input and output. We can decompose the mutual information into a sum of entropies.

$$MI(I;O) = H(I) + H(O) - H(I,O) \tag{6}$$

Where **I** and **O** are the input and output time series respectively. Initially we will take the these time series to be a vector of amplitudes separated in time, $I = (I(t_1), I(t_2)), \ldots, I(t_N)$. Latter we will take the limit of infinite time points for continuous signals.

Let's first examine how to calculate the entropy of the input distribution. The entropy of the input signal is defined in the usual way.

$$H(I) = -\sum_{\mathbf{I}} p_{\mathbf{I}}(I) \ln p_{\mathbf{I}}(I) \tag{7}$$

For Gaussian signals we define the distribution for an particular input as,

$$p_{\mathbf{I}}(I) = (2\pi)^{-N/2} |A|^{-1/2} \exp\left[-(1/2)\mathbf{I}^T \cdot A^{-1} \cdot \mathbf{I}\right] \tag{8}$$

Where $A_{j,k} = \langle I(t_j), I(t_k) \rangle$ is the correlation of the input signal between different time points. For continuous signals, this correlation can be represented by a function of the form $\langle I(t_j), I(t_k) \rangle = S(t_j - t_k)$. This follows from the assumption that the signal is time translation invariant.

Note: The notation $\langle \cdot \rangle$ means the ensemble average. In other works, take the average over multiple time series. If our system is ergodic we can define our averages as

$$\langle I(t)I(t+\tau) \rangle = ms - \lim_{T\to\infty} \int_0^T dt I(t) I(t+\tau) \tag{9}$$

Next, we want to calculate an entropy density or how much information each entry in our time series tells us.

$$H(\mathbf{I})/N = \int \mathcal{D}I \left(\frac{1}{2}\ln 2\pi + \frac{1}{2N}\ln|A| + \frac{1}{2N} I^T \cdot A^{-1} \cdot I\right) p_{\mathbf{I}}(I) \tag{10}$$

$$H(\mathbf{I})/N = (1 + \ln 2\pi + \overline{\ln \lambda^{\mathbf{I}}})/2 \tag{11}$$

Similarly, it follows the entropy density for both the output distribution and the joint input/output distribution as,

$$H(\mathbf{O})/N = (1 + \ln 2\pi + \overline{\ln \lambda^{\mathbf{O}}})/2 \tag{12}$$

$$H(\mathbf{I}, \mathbf{O})/N = 1 + \ln 2\pi + \overline{\ln \lambda^{\mathbf{I},\mathbf{O}}} \tag{13}$$

Where $\overline{\ln \lambda^{\alpha}}$ is the average of the log of eigenvalues. It follows that the mutual information between the input and out put is

$$MI(I:O)/N = \left(\overline{\ln \lambda^{\mathbf{I}}} + \overline{\ln \lambda^{\mathbf{O}}}\right)/2 - \overline{\ln \lambda^{\mathbf{I},\mathbf{O}}} \tag{14}$$

Under the assumption that the signal is time translation invariant the Fourier transform of the eigenvalues of the correlation matrix takes the form $\lambda_j^{\mathbf{I}} = \sum_k S_{I,I}(k) \exp(ik\omega_j) \equiv C_{I,I}(\omega_j)$. Similarly, the output spectrum is defined as $C_{O,O}(\omega_j)$ where $\omega_j = 2\pi j/N$.

We can calculate $\overline{\ln \lambda^{\mathbf{I}}}/\Delta t$ by taking the limit of infinite time samples.

$$\overline{\ln \lambda^{\mathbf{I}}}/\Delta t = \sum_j \ln C_{I,I}(\omega_j)/(N\Delta t) \to \int \frac{d\omega}{2\pi} \ln C_{I,I}(\omega) \tag{15}$$

Where $\frac{1}{N\Delta t} \to \frac{d\omega}{2\pi}$.

We can defined the joint probability distribution between the input and output signals as

$$p_{\mathbf{I},\mathbf{O}}(\mathbf{V} = (\mathbf{I}, \mathbf{O})) = (2\pi)^{-N/2} |Z|^{-1/2} \exp\left[-(1/2)\mathbf{V}^T \cdot Z^{-1} \cdot \mathbf{V}\right] \tag{16}$$

where,

$$Z = \begin{bmatrix} C^{\mathbf{II}}(\omega) & C^{\mathbf{IO}}(\omega) \\ C^{\mathbf{OI}}(\omega) & C^{\mathbf{OO}}(\omega) \end{bmatrix} \tag{17}$$

Where $C_{\alpha,\beta}(\omega)$ is the correlation spectral density between two time signals α and β. By taking a Fourier transform of this matrix the problem of finding the eigenvalues of a $2N \times 2N$ matrix becomes a problem of finding the solutions to a set of linear equations.

$$C_{I,I}(\omega)I(\omega) + C_{I,O}(\omega)I(\omega) = \lambda(\omega)I(\omega) \tag{18}$$
$$C_{O,I}(\omega)I(\omega) + C_{O,O}(\omega)I(\omega) = \lambda(\omega)O(\omega) \tag{19}$$

This system of equations has solutions

$$\lambda_{\pm} = (C_{I,I}(\omega) + C_{O,O}(\omega) \pm \sqrt{D})/2 \tag{20}$$
$$D = (C_{I,I}(\omega) - C_{OO}(\omega))^2 + 4|C_{IO}(\omega)|^2 \tag{21}$$

Putting it all together

$$\begin{aligned} MI(I,O)/N\Delta t &= \frac{1}{4\pi}\int d\omega \ln C_{I,I}(\omega) + \frac{1}{4\pi}\int d\omega \ln C_{O,O}(\omega) - \frac{1}{4\pi}\int d\omega \ln\left(\lambda_+(\omega)\right) - \frac{1}{4\pi}\int d\omega \ln\left(\lambda_-(\omega)\right) \\ &= \frac{1}{4\pi}\int d\omega \ln C_{I,I}(\omega) + \frac{1}{4\pi}\int d\omega \ln C_{O,O}(\omega) - \frac{1}{4\pi}\int d\omega \ln\left(C_{I,I}(\omega)C_{O,O}(\omega) - |C_{I,O}(\omega)|^2\right) \\ &= -\frac{1}{4\pi}\int d\omega \ln\left(1 - |C_{I,O}(\omega)|^2/C_{I,I}(\omega)C_{O,O}(\omega)\right) \end{aligned} \tag{22}$$

4 Multiple Inputs and Outputs

We will now study the case where we have one input and multiple out puts. Using the analysis above

$$H(\mathbf{I})/N = \left(M + M\ln 2\pi + \overline{\ln \lambda^{\mathbf{I}}}\right)/2 \tag{23}$$

$$H(\mathbf{O})/N = \left(Q + Q\ln 2\pi + \overline{\ln \lambda^{\mathbf{O}}}\right)/2 \tag{24}$$

$$H(\mathbf{I},\mathbf{O})/N = \left(Q + M + (Q+M)\ln 2\pi + \overline{\ln \lambda^{\mathbf{I,O}}}\right)/2 \tag{25}$$

Where M is the number of input signals and Q is the number of outputs. We are assuming that every discrete time series has the same number of time points, N.

$$MI(\mathbf{I}:\mathbf{O})/N = \left(\overline{\ln \lambda^{\mathbf{I}}} + \overline{\ln \lambda^{\mathbf{O}}} - \overline{\ln \lambda^{\mathbf{I,O}}}\right)/2 \tag{26}$$

The correlation matrix A now takes the form

$$A = \begin{bmatrix} C^{\mathbf{I_1I_1}} & \dots & C^{\mathbf{I_1I_M}} \\ \vdots & \ddots & \vdots \\ C^{\mathbf{I_2I_1}} & \dots & C^{\mathbf{I_MI_M}} \end{bmatrix} \tag{27}$$

In order to calculate $\overline{\ln \lambda^{\mathbf{I}}}$ we can use the technique above my transforming this matrix into its Fourier representation. We then need to solve the system of equations

$$\begin{gathered} C_{I_1,I_1}(\omega)I_1(\omega) + C_{I_1,I_2}(\omega)I_1(\omega) + \dots = \lambda(\omega)I_1(\omega) \\ C_{I_2,I_1}(\omega)I_2(\omega) + C_{I_2,I_2}(\omega)I_2(\omega) + \dots = \lambda(\omega)I_2(\omega) \\ \vdots \\ C_{I_M,I_1}(\omega)I_M(\omega) + C_{I_M,I_2}(\omega)I_M(\omega) + \dots = \lambda(\omega)I_M(\omega) \end{gathered}$$

Finding the M $\lambda(\omega)$'s is non-trivial. However, in the limit of continuous signal

$$\overline{\ln \lambda^{\mathbf{I}}(\omega)}/\Delta t = \sum_{k}^{M} \int \frac{d\omega}{2\pi} \ln \lambda_k^{\mathbf{I}}(\omega) = \int \frac{d\omega}{2\pi} \ln \prod_{k}^{M} \lambda_k^{\mathbf{I}}(\omega) = \int \frac{d\omega}{2\pi} \ln \det \widetilde{A}(\omega) \tag{28}$$

Where $\widetilde{A}(\omega)$ is the matrix representation of the system of equations defined above. Similarly, we can define $\widetilde{B}(\omega)$ as the matrix that represents the spectral densities of the output distributions and $\widetilde{Z}(\omega)$ as the full correlation matrix.

Putting everything together, we find that the information rate for M inputs and Q outputs is

$$m(I;O) = -\frac{1}{4\pi} \int d\omega \ln \left(\frac{\det \widetilde{Z}(\omega)}{\det \widetilde{A}(\omega) \det \widetilde{B}(\omega)} \right) \tag{29}$$

References

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[2] Filipe Tostevin and Pieter Rein ten Wolde. “Mutual information between input and output trajectories of Biochemical Networks”. In: *Physical Review Letters* 102.21 (2009). DOI: 10.1103/physrevlett.102.218101.